1. Convolution product of Cauchy a) Power Series b) Product of 2 Series c) Product of 2 Polynoms

1.a) Power Series

$$\sum_{k=0}^{\infty} a_k x^k \sum_{j=0}^{\infty} b_j x^j = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_k b_j x^{\overbrace{k+j}^{m}} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_k b_{\underbrace{m-k}_{j}} x^m$$

(through variable renaming) $\Updownarrow$ $\sum_{m=0}^{\infty} a_m x^m \sum_{k=0}^{\infty} b_k x^k$

$\begin{cases} m=k+j, \ 0 \le k \le m, \ 0 \le j \le m \\ j = m-k, \ k = m-j, \ j \ge 0 \Rightarrow k = m; \ j = m \Rightarrow k = 0 \end{cases} \Rightarrow 0 \le k \le m$

1.b) for $x=1$, 1.a gives (also with variable renaming and substitution)

$$\sum_{m=0}^{\infty} a_m x^m \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} a_k 1^k \sum_{j=0}^{\infty} b_j 1^j = \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_k b_{m-k} 1^m = \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_k b_{m-k}$$

1.c) Polynoms (also by variable renaming)

$$\sum_{k=m_0}^{p} a_k x^k \sum_{j=k_0}^{q} b_j x^j = \sum_{k=m_0}^{p} \sum_{j=k_0}^{q} a_k b_j x^{\overbrace{k+j}^{m}} = \boxed{\sum_{m=m_0+k_0}^{p+q} \sum_{k=\max(m_0, m-q)}^{\min(p, m-k_0)} a_k b_{m-k} x^m}$$

$\Updownarrow$ $\sum_{m=m_0}^{p} a_m x^m \sum_{k=k_0}^{q} b_k x^k$, $m_0 \le m \le p$, $k_0 \le k \le q$; $k \in \mathbb{Z}, j \in \mathbb{Z}$

For $\begin{cases} m_0 = 0 \\ k_0 = 0 \end{cases} \Rightarrow \boxed{\sum_{m=0}^{p} a_m x^m \sum_{k=0}^{q} b_k x^k = \sum_{m=0}^{p+q} \sum_{k=\max(0, m-q)}^{\min(p, m)} a_k b_{m-k} x^m}$ (normal situation)

$$\begin{array}{l} m_0 \le k \le p \\ + \ k_0 \le j \le q \\ \hline m_0 + k_0 \le k+j \le p+q \\ m_0 + k_0 \le m \le p+q \end{array}, \quad \begin{array}{l} m = k+j \\ j = m-k \\ k = m-j \end{array} \ / \begin{cases} m_0 \le k \le p \\ k_0 \le m-k \le q \\ k, m, j \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} m_0 \le k \le p \\ -q \le k-m \le -k_0 \\ k, m, j \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} m_0 \le k \le p \\ m-q \le k \le m-k_0 \\ k \in \mathbb{Z} \end{cases} \Rightarrow \max(m_0, m-q) \le k \le \min(p, m-k_0)$$

(because outside the range of definition of each polynom, each term equals zero. An example helps to see this.)

$$\lim_{\substack{p \to +\infty \\ q \to +\infty}} \sum_{m=m_0}^{p} a_m x^m \sum_{k=k_0}^{q} b_k x^k = \lim_{\substack{p \to +\infty \\ q \to +\infty}} \sum_{\substack{m=m_0+k_0 \\ m \le p+q}}^{\lim p+q} \ \sum_{k=\lim_{q \to +\infty} \max(m_0, m-q)}^{\lim_{p \to +\infty} \min(p, m-k_0)} a_k b_{m-k} x^m$$

$m_0 \le m \le p$, $m \in \mathbb{Z}$; $k_0 \le k \le p$, $k \in \mathbb{Z}$

Because $\begin{cases} m \le p \\ m \le q \\ q \to +\infty \\ p \to +\infty \end{cases}$, we have $\boxed{\sum_{m=m_0}^{+\infty (p \to +\infty)} a_m x^m \sum_{k=k_0}^{+\infty (q \to +\infty)} b_k x^k = \sum_{m=m_0+k_0}^{+\infty} \sum_{k=m_0}^{m-k_0} a_k b_{m-k} x^m}$

(because we take 1st $\begin{cases} p \to +\infty \\ q \to +\infty \end{cases}$, while m may still have finite values)

If $\begin{cases} m_0 = 0 \\ k_0 = 0 \end{cases}$, then $\lim_{\substack{p \to +\infty \\ q \to +\infty}} \sum_{\substack{m=m_0 \\ (m_0=0)}}^{p} a_m x^m \sum_{\substack{k=k_0 \\ k_0=0}}^{q} b_k x^k = \boxed{\sum_{m=0}^{+\infty} a_m x^m \sum_{k=0}^{+\infty} b_k x^k = \sum_{m=0}^{+\infty} \sum_{k=0}^{m} a_k b_{m-k} x^m}$ (Product of 2 Series as Convolution Product of Cauchy for Power Series)

Test 1) $A(x) = a_0 x^0 + a_1 x^1$ $B(x) = b_0 x^0 + b_1 x^1 + b_2 x^2$ $A(x) B(x) = (a_0 x^0 + a_1 x^1)(b_0 x^0 + b_1 x^1 + b_2 x^2) = a_0 b_0 x^0 + a_0 b_1 x^1 + a_0 b_2 x^2 + a_1 b_0 x^1 + a_1 b_1 x^2 + a_1 b_2 x^3 \Leftrightarrow$

(1): $A(x) B(x) = a_0 b_0 x^0 + (a_0 b_1 + a_1 b_0) x^1 + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + a_1 b_2 x^3$; $A(x) B(x) = \sum_{\substack{m=0 \\ m_0}}^{1=p} a_m x^m \sum_{\substack{k=0 \\ k_0}}^{2=q} b_k x^k = \sum_{\substack{m=0+0 \\ m_0+k_0}}^{3=p+q} \sum_{k=\max(0, m-2)}^{\min(1, m-0)} a_k b_{m-k} x^m = \sum_{k=0}^{0 \, (m=0)} a_k b_{m-k} x^0 + \sum_{k=0}^{1 \, (m=1)} a_k b_{m-k} x^1 + \sum_{k=0}^{1 \, (m=2)} a_k b_{m-k} x^2 + \sum_{k=1}^{(m=3)} a_k b_{m-k} x^3 \Rightarrow$

(2) $A(x) B(x) = \overbrace{a_0 b_0}^{(m=0)} x^0 + (\overbrace{a_0 b_1 + a_1 b_0}^{(m=1)}) x^1 + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + a_1 b_2 x^3 \Rightarrow (1) = (2)$ ∎

Test 2) $A(x) B(x) = (\overbrace{\sum_{m=2}^{4} a_m x^m}^{A(x)})(\overbrace{\sum_{k=3}^{5} b_k x^k}^{B(x)}) = (\sum_{\substack{m=m_0 \\ m_0=2}}^{p=4} a_m x^m)(\sum_{\substack{k=k_0 \\ k_0=3}}^{q=5} b_k x^k) = \sum_{\substack{m=m_0+k_0 \\ m=5}}^{p+q=9} \sum_{k=\max(m_0, m-q)}^{\min(p, m-k_0)} a_k b_{m-k} x^m = \sum_{k=2}^{2 \, (m=5)} a_k b_{m-k} x^5 + \sum_{k=2}^{3 \, (m=6)} a_k b_{m-k} x^6 + \sum_{k=2}^{4 \, (m=7)} a_k b_{m-k} x^7 + \sum_{k=3}^{4 \, (m=8)} a_k b_{m-k} x^8 + \sum_{k=4}^{4} a_k b_{m-k} x^9 \Rightarrow$

(1) $A(x) B(x) = a_2 b_3 x^5 + (a_2 b_4 + a_3 b_3) x^6 + (a_2 b_5 + a_3 b_4 + a_4 b_3) x^7 + (a_3 b_5 + a_4 b_4) x^8 + a_4 b_5 x^9$. Through multiplication of polynoms (distrib.) we get also:

(2) $A(x) B(x) = (a_2 x^2 + a_3 x^3 + a_4 x^4)(b_3 x^3 + b_4 x^4 + b_5 x^5) = a_2 b_3 x^5 + a_2 b_4 x^6 + a_2 b_5 x^7 + a_3 b_3 x^6 + a_3 b_4 x^7 + a_3 b_5 x^8 + a_4 b_3 x^7 + a_4 b_4 x^8 + a_4 b_5 x^9 \Rightarrow$

$A(x) B(x) = a_2 b_3 x^5 + (a_2 b_4 + a_3 b_3) x^6 + (a_2 b_5 + a_3 b_4 + a_4 b_3) x^7 + (a_3 b_5 + a_4 b_4) x^8 + a_4 b_5 x^9 \Rightarrow (1) = (2)$ ∎

Tests / Exercises for Series (Cauchy Product)

$$\underbrace{\sum_{m=m_0}^{+\infty} a_m x^m}_{A(x)} \underbrace{\sum_{k=k_0}^{+\infty} b_k x^k}_{B(x)} = \sum_{m=m_0+k_0}^{+\infty} \sum_{k=m_0}^{m-k_0} a_k b_{m-k} x^m = \underbrace{\left(a_{m_0}x^{m_0} + a_{m_0+1}x^{m_0+1} + \dots + a_m x^m + \dots\right)}_{A(x)} \underbrace{\left(b_{k_0}x^{k_0} + b_{k_0+1}x^{k_0+1} + \dots + b_k x^k + \dots\right)}_{B(x)}$$

2. Compute Product of: 2.1. $A(x).B(x) = ?$ With $A(x) = \sum_{m=2}^{+\infty} x^m$ and $B(x) = \sum_{k=1}^{\infty} (-x)^k$; 2.2. $A^3(x) = A(x)B(x)C(x)$, with $A(x) = \sum_{m=1}^{\infty} (-x)^m$

We can easily control results, because in 2.1, ($|x|<1$)

$$A(x) = \sum_{m=2}^{\infty} x^m = \sum_{m=0}^{\infty} x^m - x^1 - x^0 = \frac{1}{1-x} - \frac{x-1}{(1-x)} = \frac{1-(x+1)(1-x)}{1-x} = \frac{\cancel{1} + x^2 - \cancel{1}}{1-x} = \frac{x^2}{1-x}$$ and $$B(x) = \sum_{m=0}^{+\infty} (-x)^m - (-x)^0 = \frac{1}{1+x} - 1 = \frac{\cancel{1} - \cancel{1} - x}{1+x} = \frac{-x}{1+x}, \text{ for } \begin{cases} |-x|<1 \\ |x|<1. \end{cases}$$

$\Rightarrow |A(x)||B(x)| = \left|\frac{x^2}{1-x}\right| \cdot \left|\frac{-x}{1+x}\right| \leq \left|\frac{-x^3}{1-x^2}\right|$. The series converge absolutely within convergence ray, then $A(x)B(x) = -\frac{x^3}{1-x^2}$ (1), within conv. ray.

2.1. $$A(x)B(x) = \sum_{m=2}^{+\infty} x^m \sum_{k=1}^{\infty} (-x)^k = \sum_{m=2}^{+\infty} \underbrace{1}_{a_m} x^m \sum_{k=1}^{+\infty} \underbrace{(-1)^k}_{b_k} x^k = \sum_{\substack{m=3 \\ m_0+k_0}}^{+\infty} \sum_{\substack{k=2 \\ m_0}}^{m-1 \leftarrow k_0} \overbrace{1}^{a_k} \overbrace{(-1)^{m-k}}^{b_{m-k}} x^m = \sum_{m=3}^{+\infty} (-1)^m x^m \sum_{k=2}^{m-1} (-1)^{-k} = \sum_{m=3}^{+\infty} (-x)^m \sum_{k=2}^{m-1} \left[\frac{1}{(-1)}\right]^k = \sum_{m=3}^{+\infty} (-x)^m \cdot \left((-1)^2 \cdot \frac{1-(-1)^{m-1-2+1}}{1-(-1)}\right) \overset{(*)}{=} \sum_{m=3}^{+\infty} (-x)^m \frac{(1-(-1)^m)}{2} \Leftrightarrow$$

$$A(x)B(x) = \sum_{m=0}^{+\infty} (-x)^{m+3} \frac{(1-(-1)^{m+3})}{2} = \frac{(-x)^3}{2}\left[\sum_{m=0}^{\infty} (-x)^m - \sum (-x)^m (-1)^{m+3}\right] = -\frac{x^3}{2}\left[\frac{1}{1+x} - (-1)^3 \sum_{m\geq 0} (-x)^m (-1)^m\right] = -\frac{x^2}{2}\left[\frac{1}{1+x} + \sum_{m=0}^{\infty} (-1)^m \cdot (-1)^m x^m\right] = -\frac{x^3}{2}\left(\frac{1}{1+x} + \sum_{m=0}^{\infty} [(-1)^m]^2 x^m\right) \Leftrightarrow$$

$$A(x)B(x) = -\frac{x^3}{2}\left(\frac{1}{1+x}_{(1-x)} + \frac{1}{1-x}_{(1+x)}\right) = -\frac{x^3}{2}\left(\frac{1-\cancel{x}+1+\cancel{x}}{1-x^2}\right) = \frac{-x^3}{1-x^2} \quad (2)$$ $\Rightarrow$ (1) = (2) within convergency ray. From expression above, $A(x)B(x) \overset{(*)}{=} \sum_{m=3}^{+\infty} (-x)^m \left(\frac{1-(-1)^m}{2}\right)$,

we conclude that $A(x)B(x) = -\frac{x^3}{1-x^2} = \sum_{m=3}^{+\infty} \overbrace{\frac{(-1)^m(1-(-1)^m)}{2}}^{u_m} x^m$, and may generate u_m, where $u_0 = u_1 = u_2 = 0$, and $u_m = \begin{cases} \frac{(-1)^m(1-(-1)^m)}{2}, & \text{for } m \geq 3 \\ 0, & \text{for } m \in \{0,1,2\}. \end{cases}$

2.2. We ignore here the convergency ray details for this purpose. $A(x) = \sum_{m=1}^{\infty} (-x)^m = \frac{1}{1+x} - 1 = \frac{-x}{1+x} \Rightarrow [A(x)]^3 = \left(\frac{-x}{1+x}\right)^3 = \frac{-x^3}{(1+x)^3}$ (3)

$$A(x)B(x)C(x) = [A(x)]^3 = \Big(\overbrace{\sum_{m=1}^{\infty} \underbrace{(-1)^m}_{a_m} x^m}^{A(x)}\Big)\Big(\overbrace{\sum_{k=1}^{\infty} \underbrace{(-1)^k}_{b_k} x^k}^{B(x)}\Big)\Big(\overbrace{\sum_{j=1}^{\infty} \underbrace{(-1)^j}_{c_j} x^j}^{C(x)}\Big) = \Big(\sum_{m=2}^{+\infty} \sum_{k=1}^{m-1} \underbrace{(-1)^k}_{a_k} \underbrace{(-1)^{m-k}}_{b_{m-k}} x^m\Big) C(x) = \Big(\sum_{m=2}^{+\infty} \sum_{k=1}^{m-1} (-x)^m\Big) C(x) = \Big(\sum_{m=2}^{+\infty} (-x)^1 \Big\{\frac{1-(-x)^{m-1}}{1-(-x)}\Big\}\Big) C(x)$$

but also

$$[A(x)]^3 = \Big(\overbrace{\sum_{m=2}^{+\infty} d_m x^m}^{D(x)}\Big)\Big(\overbrace{\sum_{j=1}^{\infty} (-1)^j x^j}^{C(x)}\Big) = \sum_{m=3}^{\infty} \sum_{j=2}^{m-1} d_j c_{m-j} x^m = \sum_{m=3}^{\infty} \sum_{j=2}^{m-1} \Big(\underbrace{\sum_{k=1}^{j-1} (-1)^j}_{d_j}\Big) \underbrace{(-1)^{m-j}}_{c_{m-j}} x^m = \sum_{m=3}^{\infty} \sum_{j=2}^{m-1} \sum_{k=1}^{j-1} (-x)^m = \sum_{m=3}^{\infty} (-x)^m \sum_{j=2}^{m-1} \sum_{k=1}^{j-1} 1 = \sum_{m=3}^{\infty} (-x)^m \overbrace{\sum_{j=2}^{m-1} (j-1-1+1)}^{e_m} = \sum_{m=3}^{\infty} (-x)^m \sum_{j=1}^{m-2} j \Leftrightarrow$$

$D(x) = A(x)B(x)$; $\sum_{m=3}^{\infty} e_m x^m = [A(x)]^3$

$$[A(x)]^3 = \sum_{m=3}^{\infty} (-x)^m \left(\frac{1+m-2}{2}\right)(m-2) = \sum_{m=3}^{\infty} (-x)^m \frac{(m-1)(m-2)}{2} = \frac{(-x)^3}{2} \sum_{m=3}^{+\infty} (-x)^{m-3} (m-2)(m-1) = \frac{(-x)^3}{2} \sum_{m=3}^{+\infty} \left[(-x)^{m-1}\right]'' = -\frac{x^3}{2} \frac{d^2}{dx^2}\left[\sum_{m=3}^{+\infty} (-x)^{m-1}\right] = -\frac{x^3}{2} \frac{d^2}{dx^2}\left[\sum_{m=2}^{\infty} (-x)^m\right] = -\frac{x^3}{2} \frac{d^2}{dx^2}\left[\frac{1}{1+x} - \left[(-x)^1 + (-x)^0\right]\right] \Leftrightarrow$$

$$[A(x)]^3 = -\frac{x^3}{2} \frac{d^2}{dx^2}\left[\frac{1}{1+x} + \frac{(x-1)}{(1+x)}\right] = -\frac{x^3}{2} \frac{d^2}{dx^2}\left[\frac{\cancel{1} + x^2 - \cancel{1}}{1+x}\right] = -\frac{x^3}{2} \frac{d}{dx}\left[\frac{2x(1+x) - x^2}{(1+x)^2}\right] = -\frac{x^3}{2} \frac{d}{dx}\left[\frac{2x + x^2}{(1+x)^2}\right] = -\frac{x^3}{2}\left[\frac{(2+2x)(1+x)^2 - 2(\cancel{1+x})(2x+x^2)}{(\cancel{1+x})(1+x)^3}\right] \Leftrightarrow$$

$$[A(x)]^3 = -\frac{x^3}{\cancel{2}}\left[\frac{\cancel{2}(1+x)^2 - \cancel{2}x(2+x)}{(1+x)^3}\right] = \frac{-x^3(1 + \cancel{2x} + \cancel{x^2} - \cancel{2x} - \cancel{x^2})}{(1+x)^3} = \frac{-x^3}{(1+x)^3} \quad (4)$$, (3) = (4), thus the resolution (Cauchy calculus for series) is OK ∎