Exercise. Proof that ${}_0F_1\left(;\frac{1}{2};-\frac{z^2}{4}\right)=\cos z$, using identity ${}_pF_q\left[\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\dots(a_p)_k}{(b_1)_k(b_2)_k\dots(b_q)_k}\frac{z^k}{k!}$, where $(a_i)_k$ identifies the rising factorial as follows $(a_i)_k=a_i^{\bar{k}}=\begin{cases}1, & k=0\\ \underbrace{(a_i)(a_i+1)\dots(a_i+k-1)}_{k\text{ factors}}, & k\in N^+\end{cases}$, and ${}_pF_q\left[\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};z\right]={}_pF_q(a_1,\dots,a_p;b_1,\dots b_q;z)$

(reference: Wikipedia, generalized hypergeometric function

$$\overset{p=0\Rightarrow(a_p)_k=1}{{}_0F_1}\Big(\emptyset;\underbrace{\tfrac{1}{2}}_{q=1\Rightarrow b_q=b_1=\frac{1}{2}};\overbrace{-\tfrac{z^2}{4}}^{f(z)}\Big)=\sum_{k=0}^{\infty}\frac{1}{(b_1)_k}\frac{[f(z)]^k}{k!}=\sum_{k=0}^{\infty}\frac{1}{\prod_{i=1}^{k}(b_1+i-1)}\cdot\left(-\frac{z^2}{4}\right)^k\cdot\frac{1}{k!}=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{4^k\cdot k!\prod_{i=1}^{k}\left(\frac{1}{2}+i-\frac{2}{2}\right)}=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{2^k\cdot 2^k\cdot k!\underbrace{\prod_{i=1}^{k}\left(i-\frac{1}{2}\right)}_{\frac{1}{2}\frac{3}{2}\frac{5}{2}\dots\left(\frac{2k-1}{2}\right)=\frac{1}{2^k}\prod_{i=1}^{k}(2i-1)}}\Rightarrow$$

$${}_0F_1\left(;\frac{1}{2};-\frac{z^2}{4}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{\underbrace{2.2\dots2}_{k\text{ factors}}\cdot\underbrace{1.2.3\dots k}_{k\text{ factors}}\cdot 2^k\cdot\underbrace{\frac{1}{2}\frac{3}{2}\cdot\frac{5}{2}\dots\frac{2k-1}{2}}_{k\text{ factors}}}=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{\cancel{2^k}\cdot\frac{1}{\cancel{2^k}}\underbrace{(1.3.5.(2k-1))(2.4.6\dots2k)}_{(2k)!}}\Rightarrow$$

$${}_0F_1\left(;\frac{1}{2};-\frac{z^2}{4}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!}=\cos z\ \blacksquare$$