Multinomium Formula Explained with Summation Details

$$(a_1 + a_2 + \dots + a_m)^n = \sum_{\sum_{i=1}^{m} k_i = n} \binom{n}{k_1, k_2, \dots, k_m} a_1^{k_1} a_2^{k_2} \dots a_m^{k_m}, \text{ where } \binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \dots k_m!}, \text{ with } \sum_{i=1}^{m} k_i = n.$$

Wikiversité (French) uses serial development of exponencials to proof the theorum of the Multinomium. However to better understand the Multinomium theorem, it worths to go a bit further in detail.

(ref. wikiversité)
$$e^{t(x_1 + x_2 + \dots + x_m)} = e^{tx_1} e^{tx_2} \dots e^{tx_m} \Rightarrow \sum_{n\geq 0}^{\infty} \frac{(t(x_1 + \dots + x_m))^n}{n!} = \left(\sum_{k_1 \geq 0}^{\infty} \frac{(tx_1)^{k_1}}{k_1!}\right)\left(\sum_{k_2 \geq 0}^{\infty} \frac{(tx_2)^{k_2}}{k_2!}\right)\dots\left(\sum_{k_m \geq 0}^{\infty} \frac{(tx_m)^{k_m}}{k_m!}\right) \Leftrightarrow \sum_{n\geq 0}^{\infty} \frac{(x_1 + \dots + x_m)^n t^n}{n!} = \sum_{n}^{\infty} \sum_{k_1 + \dots + k_m = n} \frac{x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}}{k_1! k_2! \dots k_m!} t^n \Rightarrow$$

$[t^n]$ = coefficients of t^n in both sides of last equation lead to $\frac{(x_1 + \dots + x_m)^n}{n!} = \sum_{\sum_{i=1}^{m} k_i = n} \frac{x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}}{k_1! k_2! \dots k_m!}$, where we may deduce the Multinomium formula,

$$(x_1 + \dots + x_m)^n = \sum_{k_1 + \dots + k_m = n} \binom{n}{k_1, k_2, \dots, k_m} x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}.$$

But what means exactly $\sum_{k_1 + \dots + k_m = n}$? From the demonstration above, we have already one answer, based on easy calculus over Cauchy convolution products for series.

$$\sum_{k_1 + \dots + k_m = n} (\dots) = \underbrace{\sum_{k_1 = 0}^{\infty} \sum_{k_2 = 0}^{\infty} \dots \sum_{k_m = 0}^{\infty}}_{\text{with } (k_1 + \dots + k_m = n)} (\dots)$$
, based on fact that $\left(\sum_{k_1 = 0}^{\infty} a_{k_1} x^{k_1}\right)\left(\sum_{k_2 = 0}^{\infty} b_{k_2} x^{k_2}\right) = \sum_{k_1 = 0}^{\infty} \sum_{k_2 = 0}^{\infty} a_{k_1} b_{k_2} x^{k_1 + k_2}$ (product of Convolution of Cauchy for Series, Formula 1)

But there's a second formula for the Convolution Product of Series. This formula may also be used in the Multinomium Formula, but appears in a less clearly way and it is a lot harder to demonstrate. We'll try to show these details here, because it is important to understand correctly the multisommations and multi index notations in order to be able to advance in the universitary Maths.

The second Cauchy Formula for series states that $\sum_{n\geq 0}^{\infty} a_n x^n \sum_{k=0}^{\infty} b_k x^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n$, this formula may be derived formula 1 above, where n in second member of equation 2 equals $k_1 + k_2$ from 1st formula, thus,
$$\sum_{k_1 = 0}^{\infty} a_{k_1} x^{k_1} \sum_{k_2 = 0}^{\infty} b_{k_2} x^{k_2} = \sum_{k_1 = 0}^{\infty} \sum_{k_2 = 0}^{\infty} (a_{k_1} x^{k_1})(b_{k_2} x^{k_2}) = \sum_{k_1 = 0}^{\infty} \sum_{k_2 = 0}^{\infty} a_{k_1} b_{k_2} x^{\overbrace{k_1 + k_2}^{n}}$$
, thus where $\begin{cases} k_1 + k_2 = n, & 0 \leq n < \infty, n \in \mathbb{N} \\ k_1 \leq n \\ k_2 \leq n \end{cases} \Rightarrow k_2 = n - k_1, k = k_1 \Rightarrow k_2 = n - k \Rightarrow$

$$\sum_{k_1 = 0}^{\infty} a_{k_1} x^{k_1} \sum_{k_2 = 0}^{\infty} b_{k_2} x^{k_2} = \sum_{\substack{k_1 + k_2 = n \\ n = 0}}^{\infty} a_{k_1} \underbrace{b_{n-k_1}}_{k_2} x^{n} \ (n = k_1 + k_2) = \sum_{n\geq 0}^{\infty} \sum_{k_1 = 0}^{n} a_{k_1} b_{n-k_1} x^n$$

We're going to use the exponential equalities from above to deduce another equality for the Multisom notation $\sum_{k_1 + \dots + k_m = n}$

1) $m=2 \Rightarrow$ formula of Binomium

$$e^{t(x_1+x_2)} = e^{tx_1}\cdot e^{tx_2} \Rightarrow \sum_{m=0}^{\infty}\frac{[t(x_1+x_2)]^m}{m!} = \overbrace{\left(\sum_{k_1=0}^{\infty}\frac{(tx_1)^{k_1}}{k_1!}\right)}^{\sum a_m x^m}\cdot\overbrace{\left(\sum_{k_2=0}^{\infty}\frac{(tx_2)^{k_2}}{k_2!}\right)}^{\sum b_k x^k} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{x_1^{k_1}x_2^{k_2}}{k_1!\,k_2!}t^m = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{x_1^{k}}{k!}\frac{x_2^{m-k}}{(m-k)!}t^m$$

$(k_1+k_2=m)$

$$\Rightarrow [t^m] = \frac{(x_1+x_2)^m}{m!} = \sum_{k=0}^{m}\frac{m!}{k!(m-k)!}\cdot\frac{x_1^k x_2^{m-k}}{m!} \Rightarrow m![t^m] = (x_1+x_2)^m = \sum_{\substack{k=0\\(k_1=k)}}^{m}\binom{m}{\underbrace{k}_{k_1},\underbrace{m-k}_{k_2}}x_1^k x_2^{m-k} = \sum_{\substack{k_1=0\\k_1+k_2=m\\k_2=m-k}}^{m}\frac{m!}{k_1!\,k_2!}x_1^{k_1}x_2^{k_2} = \sum_{\substack{k_1=0\\k_1+k_2=m}}^{m\,(\infty)}\binom{m}{k_1,k_2}x_1^{k_1}x_2^{k_2}$$

$$(x_1+x_2)^m = \sum_{k_1+k_2=m}\binom{m}{k_1,k_2}x_1^{k_1}x_2^{k_2} = \sum_{k_1=0}^{m}\binom{m}{k_1}x_1^{k_1}x_2^{m-k_1} = \sum_{k_1=0}^{m}\binom{m}{k_1,k_2}x_1^{k_1}x_2^{k_2} = \sum_{k_1=0}^{\infty}\binom{m}{k_1,k_2}x_1^{k_1}x_2^{k_2}$$

2) $m=3 \Rightarrow$

$$e^{t(x_1+x_2+x_3)} = \prod_{i=1}^{3}e^{tx_i} \Rightarrow \sum_{m=0}^{\infty}\frac{t^m(x_1+x_2+x_3)^m}{m!} = \left(\left(\sum_{m=0}^{\infty}\frac{(tx_1)^m}{m!}\right)\left(\sum_{k_1=0}^{\infty}\frac{(tx_2)^{k_1}}{k_1!}\right)\right)\left(\sum_{k_2=0}^{\infty}\frac{(tx_3)^{k_2}}{k_2!}\right) = \left(\overbrace{\sum_{u=0}^{\infty}\sum_{k=0}^{m}\frac{(tx_1)^k}{k!}\frac{(tx_2)^{m-k}}{(m-k)!}}^{\sum_{u=0}^{\infty}u_m}\right)\left(\overbrace{\sum_{j=0}^{\infty}\frac{(tx_3)^j}{j!}}^{\sum_{j=0}^{\infty}v_j}\right) \Rightarrow$$

$$\sum_{m=0}^{\infty}\frac{t^m(x_1+x_2+x_3)^m}{m!} = \sum_{m=0}^{\infty}\sum_{j=0}^{m}u_j v_{m-j} = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\overbrace{\sum_{k=0}^{j}\frac{(tx_1)^k}{k!}\frac{(tx_2)^{j-k}}{(j-k)!}}^{u_j}\overbrace{\frac{(tx_3)^{m-j}}{(m-j)!}}^{v_{m-j}} = \sum_{m=0}^{\infty}t^m\sum_{j=0}^{m}\underbrace{\frac{x_3^{m-j}}{(m-j)!}\sum_{k=0}^{j}\frac{x_2^{j-k}}{(j-k)!}\frac{x_1^k}{k!}}$$

If $a_{32,m-j} = \frac{x_3^{m-j}}{(m-j)!}\sum_{k=0}^{j}\frac{x_2^{j-k}}{(j-k)!}\frac{x_1^k}{k!}$ through double compting we have that $\sum_{j=0}^{m}a_{m-j} = \sum_{j=0}^{m}a_j$ (compting from 0..m or compting/adding from m..0 reverse order)

$a_{32,m-j}$ = sequence a_{32} with index $m-j$

thus, $\sum_{j=0}^{m}a_{32,m-j} = \sum_{j=0}^{m}a_{32,j}$. With replacement of variables $\begin{cases} m-j \text{ into } j \\ j \text{ into } m-j \end{cases}$ we get:

$$\sum_{m=0}^{\infty}\frac{[t(x_1+x_2+x_3)]^m}{m!} = \sum_{m=0}^{\infty}t^m\sum_{j=0}^{m}\frac{x_3^{j}}{j!}\sum_{k=0}^{m-j}\frac{x_2^{m-j-k}}{(m-j-k)!}\frac{x_1^k}{k!} = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{k=0}^{m-j}\frac{x_1^k x_3^j x_2^{m-j-k}}{k!\,j!\,(m-j-k)!}\frac{t^m m!}{m!} = \sum_{m=0}^{\infty}\frac{[t(\overset{x_3}{a_1}+\overset{x_1}{a_2}+\overset{x_2}{a_3})]^m}{m!} = \sum_{m=0}^{\infty}\sum_{\substack{l_1=0\\j=l_1}}^{m}\sum_{\substack{l_2=0\\k=l_2}}^{m-l_1}\frac{a_1^{l_1}}{l_1!}\frac{a_2^{l_2}}{l_2!}\frac{a_3^{m-l_1-l_2}}{\underbrace{(m-l_1-l_2)}_{l_3}!}\cdot\frac{m!\,t^m}{m!} \Rightarrow$$

$$\sum_{m=0}^{\infty}\frac{[t(\overset{a_2}{x_1}+\overset{a_3}{x_2}+\overset{a_1}{x_3})]^m}{m!} = \sum_{m=0}^{\infty}\frac{[t(a_1+a_2+a_3)]^m}{m!} = \sum_{m=0}^{\infty}\sum_{l_1=0}^{m}\sum_{l_2=0}^{m-l_1}\frac{a_1^{l_1}a_2^{l_2}a_3^{l_3}}{m!}\binom{m}{l_1,l_2,l_3} \Rightarrow m![t^m] = (a_1+a_2+a_3)^m = \sum_{l_1=0}^{m}\sum_{l_2=0}^{m-l_1}\binom{m}{l_1,l_2,l_3}\left(\prod_{i=1}^{3}a_i^{l_i}\right)$$

$(l_1+l_2+l_3=m)$

this makes believe that $(a_1+\dots+a_m)^m = \sum_{l_1=0}^{m}\sum_{l_2=0}^{m-l_1}\dots\sum_{l_{m-1}=0}^{m-\sum_{l=1}^{m-2}l_i}\binom{m}{l_1,l_2,\dots,l_m}\prod_{i=1}^{m}a_i^{l_i} = \sum_{l_1+\dots+l_m=m}\binom{m}{l_1,l_2,\dots,l_m}\prod_{i=1}^{m}a_i^{l_i}$

With $(l_1+l_2+\dots+l_m=m)$

Based on the reasonings above, we can indeed proof this analitically, again based on exponencial equalities → Please refer to page 3

$$e^{t(x_1+\dots+x_m)} = \prod_{i=1}^{m} e^{tx_i} \Rightarrow \sum_{n=0}^{\infty} \frac{[t(x_1+\dots+x_m)]^n}{n!} = \prod_{i=1}^{m}\left(\sum_{l_i=0}^{\infty}\frac{(tx_{i-1})^{l_i}}{l_i!}\right) = \left[\left(\sum_{m=0}^{\infty}\frac{(tx_m)^m}{m!}\right)\left(\sum_{l_m=0}^{\infty}\frac{(tx_{m-1})^{l_m}}{l_m!}\right)\right]\left(\prod_{i=2}^{m-1}\left(\sum_{l_i=0}^{\infty}\frac{(tx_{i-1})^{l_i}}{l_i!}\right)\right) \Leftrightarrow$$

let $l_1 = m$, $x_{-1} = x_m$

we apply product of Cauchy for series

$$e^{t\sum_{i=1}^{m} x_i} = \prod_{i=1}^{m} e^{tx_i} \Rightarrow \sum_{n=0}^{\infty}\frac{\left(t\sum_{i=1}^{m}x_i\right)^n}{n!} = \left(\underbrace{\sum_{n=0}^{\infty}\sum_{l_m=0}^{n}\frac{(tx_m)^{l_m}}{l_m!}\cdot\frac{(tx_{m-1})^{n-l_m}}{(n-l_m)!}}_{\text{sequence } u_n = a_{mm,n}}\right)\left(\sum_{l_{m-1}=0}^{\infty}\underbrace{\frac{(tx_{m-2})^{l_{m-1}}}{l_{m-1}!}}_{v = a_{m-1}}\right)\left(\prod_{i=2}^{m-2}\left(\sum_{l_i=0}^{\infty}\frac{(tx_{i-1})^{l_i}}{l_i!}\right)\right) \Leftrightarrow$$

sequence $v_k = a_{m-1,l_{m-1}}$

$$\sum_{n=0}^{\infty}u_n\sum_{k=0}^{\infty}v_k = \sum_{n=0}^{\infty}\sum_{k=0}^{n}u_k v_{n-k}$$

$u_k = a_{mm,k}$, $v_{n-k} = a_{m-1,n-l_{m-1}}$

$k = l_{m-1} \Rightarrow u_k = a_{mm,l_{m-1}}$

sequence $a_{(m-1)mm,n}$; name: $a_{(m-1)mm}$, index: n

$$e^{t\sum_{i=1}^{m}x_i} = \prod_{i=1}^{m}e^{tx_i} \Rightarrow \sum_{n=0}^{\infty}\frac{\left(t\sum_{i=1}^{m}x_i\right)^n}{n!} = \left(\sum_{n=0}^{\infty}\sum_{l_{m-1}=0}^{n}\underbrace{\sum_{l_m=0}^{l_{m-1}}\frac{(tx_m)^{l_m}}{l_m!}\cdot\frac{(tx_{m-1})^{l_{m-1}-l_m}}{(l_{m-1}-l_m)!}}_{\text{sequence } u_k = a_{mm,\underbrace{l_{m-1}}_{k}}}\cdot\underbrace{\frac{(tx_{m-2})^{n-l_{m-1}}}{(n-l_{m-1})!}}_{\text{sequence } v_{n-k} = a_{m-1,\underbrace{n-l_{m-1}}_{n-k}}}\right)\left(\prod_{i=2}^{m-2}\left(\sum_{l_i=0}^{\infty}\frac{(tx_{i-1})^{l_i}}{l_i!}\right)\right) \Leftrightarrow$$

we're going to apply progressively this reasoning to all series of the product (right side of the equation)...

"u_k", "v_{n-k}", "k" $= l_{m-2}$ now...

$$\sum_{n=0}^{\infty}\frac{\left(t\sum_{i=1}^{m}x_i\right)^n}{n!} = \left(\sum_{n=0}^{\infty}\overset{\text{“}u_n\text{”}}{\underbrace{a_{(m-1)mm}}_{\text{name sequence } a_{(m-1)mm}}}{}_{,\underset{\text{index } n}{n}}\right)\left(\sum_{l_{m-2}=0}^{\infty}\overset{\text{“}v_k\text{”}}{\underbrace{\frac{(tx_{m-3})^{l_{m-2}}}{(l_{m-2})!}}_{a_{m-2,l_{m-2}}}}\right)\left(\prod_{i=2}^{m-3}\left(\sum_{l_i=0}^{\infty}\frac{(tx_{i-1})^{l_i}}{l_i!}\right)\right) = \sum_{n=0}^{\infty}\sum_{l_{m-2}=0}^{n}a_{(m-1)mm,l_{m-2}}\cdot a_{m-2,n-l_{m-2}} \Rightarrow$$

(we apply product of Cauchy to more complex series...)

$$\sum_{n=0}^{\infty}\frac{\left(t\sum_{i=1}^{m}x_i\right)^n}{n!} = \left(\sum_{n=0}^{\infty}\sum_{l_{m-2}=0}^{n}\sum_{l_{m-1}=0}^{l_{m-2}}\sum_{l_m=0}^{l_{m-1}}\frac{(tx_m)^{l_m}}{(l_m)!}\cdot\frac{(tx_{m-1})^{l_{m-1}-l_m}}{(l_{m-1}-l_m)!}\cdot\frac{(tx_{m-2})^{l_{m-2}-l_{m-1}}}{(l_{m-2}-l_{m-1})!}\cdot\frac{(tx_{m-3})^{n-l_{m-2}}}{(n-l_{m-2})!}\right)\left(\prod_{i=2}^{m-3}\left(\sum_{l_i=0}^{\infty}\frac{(tx_{i-1})^{l_i}}{l_i}\right)\right) \Rightarrow$$

$\vdots$

$$\sum_{n=0}^{\infty}\frac{\left(t\sum_{i=1}^{m}x_i\right)^n}{n!} = \dots = \sum_{n=0}^{\infty}\sum_{l_2=0}^{n}\sum_{l_3=0}^{l_2}\cdots\sum_{l_m=0}^{l_{m-1}}\underbrace{\frac{(tx_m)^{l_m}}{(l_m)!}\cdot\frac{(tx_{m-1})^{l_{m-1}-l_m}}{(l_{m-1}-l_m)!}}_{\sum_{l_m} a_{m(m-1),l_m}}\cdot\underbrace{\frac{(tx_{m-2})^{l_{m-2}-l_{m-1}}}{(l_{m-2}-l_{m-1})!}\cdots\frac{(tx_2)^{l_2-l_3}}{(l_2-l_3)!}}_{\prod_{i=2}^{m-2}\frac{(tx_i)^{l_i-l_{i+1}}}{(l_i-l_{i+1})!}}\cdot\frac{(tx_1)^{n-l_2}}{(n-l_2)!} \Leftrightarrow$$

(we rearrange right side equaly to $e^{t\dots}$)

sequence named $a_{m(m-1)}$, with index l_m, alias, $a_{m(m-1),l_m}$

$$e^{t\sum_{i=1}^{m}x_i} = \sum_{n=0}^{\infty}\sum_{l_2=0}^{n}\sum_{l_3=0}^{l_2}\cdots\sum_{l_{m-1}=0}^{l_{m-2}}\frac{(tx_1)^{n-l_2}}{n-l_2}\cdot\prod_{j=1}^{m-2}\frac{(tx_j)^{l_j-l_{j+1}}}{(l_j-l_{j+1})!}\sum_{l_m=0}^{l_{m-1}}\frac{(tx_m)^{l_m}}{l_m!}\frac{(tx_{m-1})^{l_{m-1}-l_m}}{(l_{m-1}-l_m)!} \qquad (1) \Rightarrow$$

Sum is a commutative operation and this implies the order of the sum when adding terms is not important.
We can thus reverse the order of counting in a sum of terms and we get the same result, this means:

$$\sum_{k=0}^{m} a_k = \sum_{\substack{k\in A\\ A=\{0,1,\dots,m\}}} a_k = \sum_{k=m}^{0} a_k = \underbrace{\sum_{k=0}^{m} a_{m-k}}_{\text{same sum of terms, but counting at reverse order}}$$

(Please note that if $a_k = f(k) \Rightarrow a_{m-k} = f(m-k)$, substitutions of vars required...) We're going to apply this principle in the demonstration of the formula of the Multinomium here, with explanation over multiple summations and more complex index notations.

$$\underbrace{\sum_{\substack{l_m=0\\ k}}^{\substack{m\\ l_{m-1}}} a_{m(m-1),l_m}}_{\text{with } b_k} = \sum_{k=0}^{m} b_k = \sum_{k=0}^{m} b_{m-k} = \sum_{l_m=0}^{l_{m-1}} a_{m(m-1),\,l_m-l_{m-1}}$$

we exchange and replace here l_m by $l_{m-1}-l_m$

with b_k $\left\{\begin{array}{l} b = a_{m(m-1)} \\ k = l_m \end{array}\right.$

$l_m \leftrightarrow l_{m-1}-l_m$, watching up replacements of variables/indices in respective sequences

(from page 3)

$$\sum_{l_m=0}^{l_{m-1}} a_{m(m-1),l_m} = \sum_{l_m=0}^{l_{m-1}} \frac{(tx_m)^{l_m}}{(l_m)!}\,\frac{(tx_{m-1})^{l_{m-1}-l_m}}{(l_{m-1}-l_m)!} = \sum_{l_m=0}^{l_{m-1}} a_{m(m-1),\,l_m-l_{m-1}} = \sum_{l_m=0}^{l_{m-1}} \frac{(tx_m)^{l_{m-1}-l_m}}{(l_{m-1}-l_m)!}\,\frac{(tx_{m-1})^{l_m}}{(l_m)!}$$

We have thus:

$$e^{t(x_1+\dots+x_m)} = \underbrace{\sum_{m=0}^{\infty}\sum_{l_2=0}^{m}\sum_{l_3=0}^{l_2}\dots\sum_{l_{m-1}=0}^{l_{m-2}}}_{\sum_{0\le l_{m-1}\le l_{m-2}\le\dots\le l_2\le m\le\infty}} \frac{(tx_1)^{m-l_2}}{(m-l_2)!}\prod_{i=1}^{m-2}\frac{(tx_i)^{l_i-l_{i+1}}}{(l_i-l_{i+1})!}\sum_{l_m=0}^{l_{m-1}}\frac{(tx_m)^{l_{m-1}-l_m}}{(l_{m-1}-l_m)!}\cdot\frac{(tx_{m-1})^{l_m}}{(l_m)!} \qquad (2)$$

$$\sum_{0\le l_{m-1}\le l_{m-2}\le\dots\le l_2\le m\le\infty} = \sum_{m=0}^{\infty}\sum_{l_2=0}^{m}\sum_{l_3=0}^{l_2}\dots\sum_{l_{m-2}=0}^{l_{m-3}}\sum_{l_{m-1}=0}^{l_{m-2}}$$

$\sum_{l_m=0}^{l_{m-1}} a_{m(m-1),\,l_m-l_{m-1}} = \sum_{l_m=0}^{l_{m-1}} a_{m(m-1),\,l_m}$, as $\sum_{k=0}^{m} b_k = \sum_{k=0}^{m} b_{m-k}$

We're going to apply this reasoning progressively to each index of each summation, the next summation to be considered is $\sum_{l_{m-1}=0}^{l_{m-2}} h_{l_{m-1}} = \sum_{l_{m-1}=0}^{l_{m-2}} h_{l_{m-2}-l_{m-1}} \Rightarrow$ replacement of all terms with l_{m-1} with $l_{m-2}-l_{m-1}$, $l_{m-2}-l_{m-1}$ by l_{m-1}. Right term of (2) shall thus be rearranged.

$$e^{\sum_{i=1}^{m} tx_i} = \sum_{0\le l_{m-2}\le l_{m-3}\le\dots\le l_2\le m\le\infty} \underset{(m=l_1)}{\frac{(tx_1)^{m-l_2}}{(m-l_2)!}}\cdot\prod_{i=1}^{m-3}\frac{(tx_i)^{l_i-l_{i+1}}}{(l_i-l_{i+1})!}\cdot\underbrace{\sum_{l_{m-1}=0}^{l_{m-2}}\frac{(tx_{m-2})^{l_{m-2}-l_{m-1}}}{(l_{m-2}-l_{m-1})!}\cdot\sum_{l_m=0}^{l_{m-1}}\frac{(tx_{m-1})^{l_m}}{l_m!}\,\frac{(tx_m)^{l_{m-1}-l_m}}{(l_{m-1}-l_m)!}}_{\sum_{l_{m-1}=0}^{l_{m-2}} h_{l_{m-1}} = \sum_{l_{m-1}=0}^{l_{m-2}} h_{l_{m-2}-l_{m-1}}}$$

$$e^{\sum_{i=1}^{m} tx_i} = \sum_{0\le l_{m-2}\le\dots\le l_2\le m\le\infty}\frac{tx_1^{m-l_2}}{(m-l_2)!}\prod_{i=1}^{m-3}\frac{tx_i^{l_i-l_{i+1}}}{(l_i-l_{i+1})!}\sum_{l_{m-1}=0}^{l_{m-2}}\overbrace{\sum_{l_m=0}^{l_{m-2}-l_{m-1}}\frac{(tx_{m-2})^{l_{m-1}}}{(l_{m-1})!}\,\frac{(tx_{m-1})^{l_m}}{l_m!}\,\frac{(tx_m)^{(l_{m-2}-l_{m-1})-l_m}}{(l_{m-2}-l_{m-1}-l_m)!}}$$

proceeding this reasoning, we get, (e.g., $l_{m-2} \hookrightarrow l_{m-3}-l_{m-2}$ and so on...)

$$e^{\sum_{i=1}^{m} t x_i} = \sum_{m=0}^{\infty} \sum_{l_2=0}^{m} \cdots \sum_{l_{m-3}=0}^{l_{m-4}} \frac{(t x_1)^{m-l_2}}{(m-l_2)!} \prod_{i=1}^{m-4} \frac{(t x_i)^{l_i-l_{i+1}}}{(l_i-l_{i+1})!} \sum_{l_{m-2}=0}^{l_{m-3}} \frac{(t x_{m-3})^{l_{m-3}-l_{m-2}}}{(l_{m-3}-l_{m-2})!} \cdot \sum_{l_{m-1}=0}^{l_{m-2}} \sum_{l_m=0}^{l_{m-2}-l_{m-1}} \frac{(t x_{m-2})^{l_{m-1}}}{(l_{m-1})!} \frac{(t x_{m-1})^{l_m}}{(l_m)!} \frac{(t x_m)^{l_{m-2}-(l_{m-1}+l_m)}}{(l_{m-2}-(l_{m-1}+l_m))!} \Leftrightarrow$$

$$e^{\sum_{i=1}^{m} t x_i} = \sum_{m=0}^{\infty} \sum_{l_2=0}^{m} \sum_{l_3=0}^{l_2} \cdots \sum_{l_{m-3}=0}^{l_{m-4}} \frac{(t x_1)^{m-l_2}}{(m-l_2)!} \left(\prod_{j=1}^{m-4} \frac{(t x_j)^{l_j-l_{j+1}}}{(l_j-l_{j+1})!} \right) \sum_{l_{m-2}=0}^{l_{m-3}} \sum_{l_{m-1}=0}^{l_{m-3}-l_{m-2}} \sum_{l_m=0}^{l_{m-3}-l_{m-2}-l_{m-1}} \underbrace{\frac{(t x_{m-3})^{l_{m-2}}}{(l_{m-2})!} \frac{(t x_{m-2})^{l_{m-1}}}{(l_{m-1})!} \frac{(t x_{m-1})^{l_m}}{l_m}}_{\prod_{k=m-3}^{m-1} \frac{(t_k)^{l_{k+1}}}{(l_{k+1})!}} \frac{(t x_m)^{l_{m-3}-\sum_{i=m-2}^{m} l_i}}{(l_{m-3}-\sum_{i=m-2}^{m} l_i)!} \Leftrightarrow$$

$$e^{\sum_{i=1}^{m} t x_i} = \sum_{0 \le l_{m-3} \le \cdots \le l_2 \le m < \infty} \frac{(t x_1)^{m-l_2}}{(m-l_2)!} \prod_{j=1}^{m-4} \frac{(t x_j)^{l_j-l_{j+1}}}{(l_j-l_{j+1})!} \sum_{l_{m-2}=0}^{l_{m-3}} \sum_{l_{m-1}=0}^{l_{m-3}-l_{m-2}} \sum_{l_m=0}^{l_{m-3}-\sum_{k=m-2}^{m-1} l_k} \prod_{k=m-3}^{m-1} \frac{(t_k)^{l_{k+1}}}{(l_{k+1})!} \cdot \frac{(t x_m)^{l_{m-3}-\sum_{k=m-2}^{m} l_k}}{(l_{m-3}-\sum_{k=m-2}^{m} l_k)!} \Leftrightarrow$$

$\vdots$ (we proceed the replacement through double counting in summations and we shall get

$$e^{\sum_{i=1}^{m} t x_i} = \dots = \sum_{m=0}^{\infty} \sum_{l_2=0}^{m} \sum_{l_3=0}^{m-l_2} \sum_{l_4=0}^{m-l_2-l_3} \cdots \sum_{l_m=0}^{m-\sum_{k=2}^{m-1} l_k} \frac{m!}{m!} \frac{(t x_1)^{l_2}}{l_2!} \frac{(t x_2)^{l_3}}{l_3!} \cdots \frac{(t x_{m-1})^{l_m}}{l_m!} \underbrace{\frac{(t x_m)^{m-\sum_{i=2}^{m} l_i}}{(m-\sum_{i=2}^{m} l_i)!}}_{l_1!}$$

, with $l_1 = m - \sum_{i=2}^{m} l_i$, we have $\sum_{i=1}^{m} l_i = m$, $\Rightarrow$

$$\sum_{m=0}^{\infty} \frac{(t \sum_{i=1}^{m} x_i)^m}{m!} = \sum_{m=0}^{\infty} \sum_{l_2=0}^{m} \sum_{l_3=0}^{m-l_2} \sum_{l_4}^{m-l_2-l_3} \cdots \sum_{l_m=0}^{m-\sum_{k=2}^{m-1} l_i} \frac{1}{m!} \overbrace{\frac{m!}{l_1!\, l_2!\, l_3! \dots l_m!}}^{\binom{m}{l_1, l_2, \dots, l_m}} t^{\sum_{i=1}^{m} l_i} \left(\prod_{i=1}^{m-1} x_i^{l_{i+1}} \right) x_m^{l_1}$$

(with $\sum_{i=2}^{m} l_i = m - l_1 \Leftrightarrow \sum_{i=1}^{m} l_i = m$)

. If we replace $\begin{cases} x_1 = a_2 \\ x_2 = a_3 \\ \vdots \\ x_{m-1} = a_m \\ x_m = a_1 \end{cases}$, we get:

$$\sum_{m=0}^{\infty} \left(\sum_{i=1}^{m} a_i \right)^m \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{l_1+\dots+l_m=m} \binom{m}{l_1, l_2, \dots, l_m} \prod_{i=1}^{m} a_i^{l_i} \frac{t^m}{m!} \quad (3)$$

Getting the coefficient of the serie $\sum_{m=0}^{\infty} c_m \frac{t^m}{m!}$, $\frac{c_m}{m!} = [t^m]$ from both sides of equation (3), we get the Multinomial identity

$$[t^m] \Rightarrow \frac{(\sum_{i=1}^{m} a_i)^m}{\cancel{m!}} = \underbrace{\sum_{l_2=0}^{m} \sum_{l_3=0}^{m-l_2} \sum_{l_4}^{m-l_2-l_3} \cdots \sum_{l_m=0}^{m-\sum_{i=2}^{m-1} l_i}}_{\text{(with } l_1+l_2+\dots+l_m=m)} \binom{m}{l_1, l_2, \dots, l_m} \prod_{j=1}^{m} a_j^{l_j} \cdot \frac{1}{\cancel{m!}} \Rightarrow \boxed{(a_1+a_2+\dots+a_m)^m = \sum_{l_1+l_2+\dots+l_m=m} \binom{m}{l_1, l_2, \dots, l_m} \prod_{j=1}^{m} a_j^{l_j}}$$

Multinomium identity

$\sum_{l_1+l_2+\dots+l_m=m}$, $l_1 = m - \sum_{i=2}^{m} l_i$

$$(a_1+a_2+\dots+a_m)^n = \sum_{\sum_{i=1}^{m} l_i = n} \binom{n}{l_1, l_2, \dots, l_m} \prod_{j=1}^{m} a_j^{l_j} = \sum_{l_2=0}^{n} \sum_{l_3=0}^{n-l_2} \sum_{l_4=0}^{n-l_2-l_3} \cdots \sum_{l_m=0}^{n-\sum_{k=2}^{m-1} l_k} \frac{n!}{l_1!\, l_2! \dots l_m!} \prod_{j=1}^{m} a_j^{l_j}$$

Test with (Exercise) $\begin{cases} a_1 = 2 \\ a_2 = -1 \\ a_3 = -2 \\ m = 3 \\ n = 3 \end{cases}$, $\left(\sum_{i=1}^{m} a_i\right)^n = (a_1+a_2+a_3)^3 = (2-1-2)^3 = (-1)^3 = -1.$

$$\left(\sum_{i=1}^{m} a_i\right)^n = (a_1+a_2+a_3)^3 = \sum_{l_2=0}^{n=3} \sum_{l_3=0}^{n-l_2=3-l_2} \binom{n}{l_1,l_2,l_3} a_1^{\overbrace{l_1}^{l_1=n-l_2-l_3}} a_2^{l_2} a_3^{l_3} = \sum_{l_1+l_2+l_3=n} \binom{n}{l_1,l_2,l_3} a_1^{n-l_2-l_3} a_2^{l_2} a_3^{l_3}$$

$$(a_1+a_2+a_3)^3 = (2-1-2)^3 = -1 = \sum_{l_2=0}^{3} \sum_{l_3=0}^{3-l_2} \binom{3}{3-l_2-l_3, l_2, l_3} 2^{3-l_2-l_3} \cdot (-1)^{l_2}\, 2^{l_3}$$

$$\left(\sum_{i=1}^{3} a_i\right)^3 = \underset{(l_2=0)}{\sum_{l_3=0}^{3}} \binom{3}{3-l_3, 0, l_3} a_1^{3-l_3} a_2^0 a_3^{l_3} + \underset{(l_2=1)}{\sum_{l_3=0}^{2}} \binom{3}{2-l_3, 1, l_3} a_1^{3-\overset{l_2}{1}-l_3} a_2^1 a_3^{l_3} + \underset{l_2=2}{\sum_{l_3=0}^{3-\overset{l_2}{2}}} \binom{3}{3-2-l_3, 2, l_3} a_1^{3-2-l_3} a_2^2 a_3^{l_3} + \underset{(l_2=3)}{\sum_{l_3=0}^{0}} \binom{3}{0,3,0} a_1^0 a_2^3 a_3^0$$

$$(a_1+a_2+a_3)^3 = \underset{(l_2=0)}{\sum_{l_3=0}^{3}} \binom{3}{3-l_3, 0, l_3} a_1^{3-l_3} a_3^{l_3} + \underset{(l_2=1)}{\sum_{l_3=0}^{2}} \binom{3}{2-l_3, 1, l_3} a_1^{2-l_3} a_2 a_3^{l_3} + \underset{(l_2=2)}{\sum_{l_3=0}^{1}} \binom{3}{1-l_3, 2, l_3} a_1^{1-l_3} a_2^2 a_3^{l_3} + \underset{(l_2=3)}{\frac{3!}{3!}} a_2^3 \Rightarrow$$

$$(2-1-2)^3 = -1 = \sum_{l_3=0}^{3} \binom{3}{3-l_3, 0, l_3} a_1^{3-l_3} a_3^{l_3} + \sum_{l_3=0}^{2} \binom{3}{2-l_3, 1, l_3} a_1^{2-l_3} a_2 a_3^{l_3} + \binom{3}{\underset{l_1}{1}, \underset{l_2}{2}, \underset{l_3}{0}} a_1^{\overset{l_1}{1}} a_2^{\overset{l_2}{2}} a_3^{\overset{l_3}{0}} + \binom{3}{\underset{l_1}{0}, \underset{l_2}{2}, \underset{l_3}{1}} a_1^{\overset{l_1}{0}} a_2^{\overset{l_2}{2}} a_3^{\overset{l_3}{1}} + \underset{\substack{(l_1=l_3=0) \\ (a_2=-2)}}{a_2^{\overset{l_2}{3}}} \Rightarrow$$

$$-1 = \sum_{l_3=0}^{3} \binom{3}{3-l_3, 0, l_3} a_1^{3-l_3} a_3^{l_3} + \underset{(l_2=1)}{\sum_{l_3=0}^{2}} \binom{3}{2-l_3, 1, l_3} a_1^{2-l_3} a_2 a_3^{l_3} + \frac{3!}{1!\,2!\,0!} a_1 a_2^2 + \underset{(a_2=-1,\ a_3=-2)}{\frac{3!}{2!} a_2^2 a_3 + (-2)^3} \Rightarrow$$

$$-1 = \sum_{l_3=0}^{3} \binom{3}{3-l_3, 0, l_3} a_1^{3-l_3} a_3^{l_3} + \underset{(l_2=1,\ l_3=0)}{\binom{3}{2-0, 1, 0} a_1^{2-0} a_2^{\overset{l_2}{1}} a_3^{\overset{l_3}{0}}} + \underset{(l_2=1,\ l_3=1)}{\binom{3}{2-1, 1, 1} a_1^{2-1} a_2^1 a_3^1} + \underset{(l_2=1,\ l_3=2)}{\binom{3}{0, 1, 2} a_1^{\overset{l_1}{0}} a_2^{\overset{l_2}{1}} a_3^{\overset{l_3}{2}}} + \frac{3!}{2!} a_1 a_2^2 + 3 \cdot (-1)^2 \cdot (-2) + (\overbrace{-8}^{a_2^3}) \Rightarrow$$

$$-1 = \underset{(l_2=0)}{\sum_{l_3=0}^{3}} \binom{3}{3-l_3, 0, l_3} a_1^{3-l_3} a_3^{l_3} + \binom{3}{2,1,0} a_1^2 a_2^1 + \binom{3}{1,1,1} a_1 a_2 a_3 + \frac{3!}{2!} a_2 a_3^2 + 3 a_1 a_2^2 + (-6) + (-8) \Rightarrow$$

$$\Big(\sum_{i=1}^{3} a_i\Big)^3 = \binom{3}{3,0,0} a_1^3 a_2^0 a_3^0 + \binom{3}{2,0,1} a_1^2 a_2^0 a_3^1 + \binom{3}{1,0,2} a_1 a_3^2 + \binom{3}{0,0,3} a_1^0 a_2^0 a_3^3 + \frac{3!}{2!} a_1^2 a_2 + 3!\, a_1 a_2 a_3 + 3 a_2 a_3^2 + 3 a_1 a_2^2 + 3 a_2^2 a_3 + a_2^3 \Rightarrow$$

$(l_2=0\ l_3=0)$ $(l_2=0,\ l_3=1)$ $(l_2=0,\ l_3=2)$ $(l_2=0\ l_3=3)$

$$(a_1+a_2+a_3)^3 = \frac{3!}{3!} a_1^3 + \frac{3!}{2!} a_1^2 a_3 + \frac{3!}{2!} a_1 a_3^2 + \frac{3!}{3!} a_3^3 + 3 a_1^2 a_2 + 6 a_1 a_2 a_3 + 3 a_2 a_3^2 + 3 a_1 a_2^2 + 3 a_2^2 a_3 + a_2^3 \Leftrightarrow$$

$$\Big(\sum_{i=1}^{3} a_i\Big)^3 = a_1^3 + a_2^3 + a_3^3 + 3(a_1^2 a_3 + a_1 a_3^2 + a_1^2 a_2 + a_2 a_3^2 + a_1 a_2^2 + a_2^2 a_3) + 6 a_1 a_2 a_3 \quad (4) \Rightarrow a_1 = 2,\ a_2 = -1,\ \underbrace{a_3}_{m} = -2,\ m=3,\ n=3 \Rightarrow (a_1+a_2+a_m)^n = \underbrace{(+2-1-2)^3}_{-1}$$

$$(+2-1-2)^3 = 2^3 + (-1)^3 + (-2)^3 + 3\big(2^2(-2) + 2(-2)^2 + 2^2(-1) + (-1)(-2)^2 + 2(-1)^2 + (-1)^2(-2)\big) + 6(2)(-1)(-2) \Rightarrow$$

$-1 = \cancel{8} - 1 - \cancel{8} + 3(-\cancel{8} + \cancel{8} + (-4) + (-4) + \cancel{2} - \cancel{2}) + 24 \Leftrightarrow -1 = -1 - 24 + 24 \Rightarrow -1 = -1$ OK $\Rightarrow$ formula works good for this test.

If we apply distributivity for $\big(\sum_{i=1}^{3} a_i\big)^3 = (a_1+a_2+a_3)^3 = (a_1+a_2+a_3)(a_1+a_2+a_3)(a_1+a_2+a_3)$, we get:

$$(a_1+a_2+a_3)^3 = a_1^2(a_1+a_2+a_3) + a_1 a_2(a_1+a_2+a_3) + a_1 a_3(a_1+a_2+a_3) + a_2 a_1(a_1+a_2+a_3) + a_2^2(a_1+a_2+a_3) + a_2 a_3(a_1+a_2+a_3) +$$
$$+ a_3 a_1(a_1+a_2+a_3) + a_3 a_2(a_1+a_2+a_3) + a_3^2(a_1+a_2+a_3) \Rightarrow$$

$$(a_1+a_2+a_3)^3 = (a_1+a_2+a_3)\underbrace{(a_1^2 + a_1 a_2 + a_1 a_3 + a_2 a_1 + a_2^2 + a_2 a_3 + a_3 a_1 + a_3 a_2 + a_3^2)}_{(a_1+a_2+a_3)^2} \Rightarrow$$

$$(a_1+a_2+a_3)^3 = \overbrace{(a_1^3 + a_1^2 a_2 + a_1^2 a_3 + a_1^2 a_2 + a_1 a_2^2 + a_1 a_2 a_3 + a_1^2 a_3 + a_1 a_2 a_3 + a_1 a_3^2)}^{a_1(a_1+a_2+a_3)^2} + \overbrace{(a_2 a_1^2 + a_1 a_2^2 + a_1 a_2 a_3 + a_1 a_2^2 + a_2^3 + a_2^2 a_3 + a_1 a_2 a_3 + a_2^2 a_3 + a_2 a_3^2)}^{a_2(a_1+a_2+a_3)^2} +$$
$$+ \underbrace{(a_1^2 a_3 + a_1 a_2 a_3 + a_1 a_3^2 + a_1 a_2 a_3 + a_2^2 a_3 + a_2 a_3^2 + a_1 a_3^2 + a_2 a_3^2 + a_3^3)}_{a_3(a_1+a_2+a_3)^2} \Rightarrow$$

$$(a_1+a_2+a_3)^3 = a_1^3 + a_2^3 + a_3^3 + 3 a_1^2 a_2 + 3 a_1^2 a_3 + 3 a_1 a_2^2 + 3 a_1 a_3^2 + 3 a_2^2 a_3 + 3 a_2 a_3^2 + 6 a_1 a_2 a_3 \quad (5)$$

$$(a_1+a_2+a_3)^3 = \binom{3}{3,0,0} a_1^3 + \binom{3}{0,3,0} a_2^3 + \binom{3}{0,0,3} a_3^3 + \binom{3}{2,1,0} a_1^2 a_2 + \binom{3}{2,0,1} a_1^2 a_3 + \binom{3}{1,2,0} a_1 a_2^2 + \binom{3}{1,0,2} a_1 a_3^2 + \binom{3}{0,2,1} a_2^2 a_3 + \binom{3}{0,1,2} a_2 a_3^2 + \binom{3}{1,1,1} a_1^1 a_2^1 a_3^1 \Rightarrow$$

$$\Big(\sum_{i=1}^{\underset{3}{m}} a_i\Big)^{\underset{3}{n}} = \sum_{l_1+l_2+l_3=\underset{3}{n}} \binom{n}{l_1, l_2, l_3} a_1^{l_1} a_2^{l_2} a_3^{l_3}$$

Also (5) = (4), applying commutativity and developping product by 3. We get thus exactly the same result.